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On the asymptotics of a 1-parameter family of infinite measure preserving transformations

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—Dedicated to R. Mañé

Abstract. We estimate various aspects of the growth rates of ergodic sums for some infinite measure preserving transformations which are not rationally ergodic.

Keywords: Infinite measure preserving transformation, logarithmic ergodic theorem, asymptotics, rationally ergodic.

0. Ergodic sums of infinite measure preserving transformations

Let $T = (X_T, \mathcal{B}_T, m_T, T)$ be a conservative, ergodic measure preserving transformation of a σ-finite, infinite, nonatomic standard measure space. It is known ([**Ho**], see also [**A**],[**Kr**]) that for

$$f \in L^1(m_T)_+ := \{f \in L^1(m_T) : \ f \geq 0, \ \int_X f dm_T > 0\},$$

$$S_n(f)(x) = S_n^T(f) := \sum_{k=0}^{n-1} f(T^k x) \to \infty \text{ for a.e. } x \in X,$$

and for $f, g \in L^1_+$:

$$\frac{S_n(f)(x)}{S_n(g)(x)} \to \frac{\int_X f dm}{\int_X g dm} \text{ for a.e. } x \in X,$$

whence,

$$S_n(f) = o(n) \ \text{ a.e.}$$

On the other hand, for any sequence of constants $(a_n)_{n \in \mathbb{N}}$,

$$S_n(f) \not\asymp a_n \text{ a.e.}$$

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as was shown in [**A2**] (see also [**A**]).

A rationally ergodic transformation $T = (X_T, \mathcal{B}_T, m_T, T)$ satisfies a kind of ergodic theorem :

$$\forall\ n_k \to \infty,\ \exists\ m_\ell = n_{k_\ell} \to \infty\ \ni\ \frac{1}{N}\sum_{\ell=1}^{N}\frac{S_{m_\ell}(f)}{a_{m_\ell}} \to \int_X f dm \text{ a.e. } \forall\ f \in L^1 \quad (1)$$

where $a_n = a_n(T)$ are constants ([**A1**], see also [**A**]). This sequence of constants, called the *return sequence*, is determined by (1) uniquely up to asymptotic equality, and can therefore be considered to represent the absolute rate of growth of $S_n(f)$ as $n \to \infty$ for $f \in L^1_+$.

In order to study the rate of growth of $S_n^T(f) \to \infty$ for general T, define as in [**A3**] the *median sequences* $\alpha_n(P, f, \theta)$ for P a m_T-absolutely continuous probability on X_T, $f \in L^1(m_T)_+$, $0 < \theta < 1$ by

$$\alpha_n(P, f, \theta) := \max\{t \geq 0 :\ P([S_n(f) \geq t]) \geq \theta\}.$$

For example if $T : \mathbb{R} \to \mathbb{R}$ is Boole's transformation defined by $Tx = x - \frac{1}{x}$, then T is a conservative, ergodic, measure preserving transformation of $\mathbb{R}$ equipped with Lebesgue measure (see [**Ad-W**]) and is rationally ergodic with return sequence $a_n(T) \sim \frac{\sqrt{2n}}{\pi}$ ([**A3**], see also [**A**]).

It is also shown in [**A3**] that

$$P\Big(\Big[\sum_{k=0}^{n-1} f \circ T^k \geq \frac{\sqrt{2n}}{\pi}t\Big]\Big) \to \frac{2}{\pi}\int_t^\infty e^{-\frac{s^2}{\pi}} ds$$

as $n \to \infty$ for $t \geq 0$ and $f \in L^1_+$, $\int_X f dm = 1$; whence

$$\alpha_n(P, f, \theta) \sim \frac{\sqrt{2n}\ \eta(\theta)}{\pi}\int_X f dm$$

where

$$\frac{2}{\pi}\int_{\eta(\theta)}^\infty e^{-\frac{s^2}{\pi}} ds = \theta.$$

A different kind of behaviour is exhibited by a conservative, ergodic, measure preserving transformation $T = (X_T, \mathcal{B}_T, m_T, T)$ which is *squashable* (see [**A**]) in the sense that it commutes with a non singular transformation Q which is not measure preserving).

In this case (as shown in [**A3**]) there is no ergodic theorem of type (1), and moreover $\frac{a_n(P,f,\theta)}{a_n(Q,g,\theta')} \to 0$ as $n \to \infty$ $\forall\, P, Q$ m_T-absolutely continuous probabilities on X_T, $f, g \in L^1(m_T)_+$, $0 < \theta' < \theta < 1$.

Suppose that $R : W \to W$ is a non-singular transformation of the probability space $(W, \mathcal{B}, \mu)$ and that

$$\frac{d\mu \circ R}{d\mu} = c^{\phi}$$

where $0 < c < 1$ and $\phi : W \to \mathbb{Z}$.

The *Maharam $\mathbb{Z}$-extension* of R is the skew product transformation $T : W \times \mathbb{Z} \to W \times \mathbb{Z}$ defined by $T(x,n) = (Rx, n - \phi(x))$ considered with respect to the invariant measure m_T defined by $m_T(A \times \{n\}) = \mu(A)c^n$. The Maharam $\mathbb{Z}$-extension of R is ergodic if, and only if R is of type III_c (see [**A**], [**W**]); and in this case it is squashable commuting with the transformation $Q(x,n) = (x, n+1)$ (for which $m_T \circ Q = c m_T$).

In this paper we look at the 1-parameter family of Maharam $\mathbb{Z}$-extensions considered in [**H-I-K**] proving a logarithmic pointwise ergodic theorem as in [**Fi**] and evaluating their median sequences.

It turns out that a limiting transformation of our 1-parameter family is actually boundedly rationally ergodic with return sequence

$$a_n \asymp \frac{n}{\sqrt{\log n}}.$$

This latter phenomenology was also obtained for some analogous transformations in [**A-K**], but by rather different methods.

1. The 1-parameter family

Let $\Omega = \{0,1\}^{\mathbb{N}}$, and $\mathcal{B}$ is the σ-algebra generated by cylinders. Define the *adding machine* $\tau : \Omega \to \Omega$ by

$$\tau(1, ..., 1, 0, \epsilon_{n+1}, \epsilon_{n+2}, ...) = (0, ..., 0, 1, \epsilon_{n+1}, \epsilon_{n+2}, ...).$$

For $p \in (0,1)$, define a probability μ_p on Ω by

$$\mu_p([\epsilon_1, ..., \epsilon_n]) = \prod_{k=1}^{n} p(\epsilon_k)$$

where $p(0) = 1 - p$ and $p(1) = p$.

It is not hard to show that $\mu_p \circ \tau \sim \mu_p$, and

$$\frac{d\mu_p \circ \tau}{d\,\mu_p} = \left(\frac{1-p}{p}\right)^{\phi}$$

where

$$\phi(x) = \sum_{n=1}^{\infty}(x_n - (\tau x)_n) = \min\{n \in \mathbb{N} : x_n = 0\} - 2.$$

This means that τ is an invertible non-singular transformation of $(\Omega, \mathcal{B}, \mu_p)$ and a measure preserving transformation of $(\Omega, \mathcal{B}, \mu_{1/2})$.

It is well known that τ is ergodic on $(\Omega, \mathcal{B}, \mu_p)$, (indeed, τ-invariant sets are tail-measurable and hence trivial by the Kolmogorov $0-1$ law).

Set,

$$X = \Omega \times \mathbb{Z}, \quad T(x,n) = (\tau x, n - \phi(x)),$$

and, for $p \in (0,1)$,

$$m_p(A \times \{n\}) = \mu_p(A)\left(\frac{1-p}{p}\right)^n.$$

Our 1-parameter family is $\{T_p : p \in (0,1),\ 0 < p \le \frac{1}{2}\}$ where

$$T_p := (X, \mathcal{B}, m_p, T).$$

Even though T_p is defined for $\frac{1}{2} < p < 1$, we "stop" at $p = \frac{1}{2}$ because T_p^{-1} is isomorphic with T_{1-p} by $(x,n) \leftrightarrow (\pi x, -n)$ where $(\pi x)_n := 1 - x_n$.

As above, $m_p \circ T^{-1} = m_p$ and $TQ = QT$ where $Q(x,n) = (x, n+1)$.

It was shown in [**H-I-K**] (see also [**A**]) that T_p is ergodic $\forall\, p \in (0,1)$, whence T_p, being an ergodic Maharam $\mathbb{Z}$-extension, is squashable for $p \ne \frac{1}{2}$.

It follows from results in [**A4**] (see [**A**]) that the representation of T_p for $p \ne \frac{1}{2}$ as a Maharam $\mathbb{Z}$-extension of a transformation of type $III_{\frac{p}{1-p}}$ is unique (up to isomorphism of the type $III_{\frac{p}{1-p}}$ transformation).

2. The results

Theorem 1. *For every* $p \in (0,1)$,

$$\frac{\log S_n(f)}{\log n} \to \hat{H}(p)\ m_p\text{-a.e. } \forall\ f \in L^1_+(m_p) \tag{2}$$

where $H(p) := -p \log p - (1-p)\log(1-p)$ *and* $\hat{H}(p) := \frac{H(p)}{log 2}$.

Theorem 2. *For* $p \neq \frac{1}{2}$*:*

$$\alpha_n(P, f, \theta) = n^{\hat{H}(p)} e^{c_p \xi(\theta)\sqrt{\log n}(1+o(1))} \tag{3}$$

as $n \to \infty$ $\forall$ P *a* m_p*-absolutely continuous probability on* X, $f \in L^1(m_p)_+$ *and* $0 < \theta < 1$ *where*

$$c_p = \sqrt{\frac{p(1-p)}{\log 2}} \log \frac{1-p}{p} \quad \text{and} \quad \int_{\xi(\theta)}^{\infty} \frac{e^{-\frac{t^2}{2}}}{\sqrt{2\pi}} dt = \theta;$$

$$\varliminf_{n\to\infty} \frac{S_n(f)}{n^{\hat{H}(p)} e^{t\sqrt{\log n \log^{(3)} n}}} = \begin{cases} 0 & t > -c_p \\ \infty & t < -c_p \end{cases}$$

and

$$\varlimsup_{n\to\infty} \frac{S_n(f)}{n^{\hat{H}(p)} e^{t\sqrt{\log n \log^{(3)} n}}} = \begin{cases} 0 & t > c_p \\ \infty & t < c_p \end{cases} \tag{4}$$

a.e. $\forall$ $f \in L^1(m_p)_+$ *where* $\log^{(3)} n := \log\log\log n$.

Theorem 3. *For* $p = \frac{1}{2}$, T *is boundedly rationally ergodic, and*

$$a_n(T_{\frac{1}{2}}) \asymp \frac{n}{\sqrt{\log n}}.$$

3. The Main Lemma

For $x = (x_1, x_2, \ldots) \in \Omega$, and $n \in \mathbb{N}$, let

$$\rho_n(x) = \min\{1 \le r \le n : x_{n-r} = 0\}, \qquad \sigma_n(x) = \min\{s \ge 1 : x_{n+s} = 0\},$$

$$s_n(x) = \sum_{k=1}^{n} x_k, \; p_n = \frac{s_n}{n}, \quad N_n(x) = S_{2^n}(1_{\Omega\times\{0\}})(x, 0).$$

Note that

$$s_n \sim np, \quad \& \; \limsup_{n\to\infty} \frac{\rho_n}{\log n} = \limsup_{n\to\infty} \frac{\sigma_n}{\log n} = \frac{1}{\log \frac{1}{p}} \qquad \mu_p - \text{a.e.}.$$

Main Lemma.

$$N_n(x) = \Phi_n(x) \binom{n}{s_n(x)}$$

where

$$|\log \Phi_n| = O(\log n) \quad \mu_p - \quad a.e.,$$

and

$$\forall\ \epsilon > 0\ \exists\ M = M_\epsilon,\ n_\epsilon\ \ni\ \mu_p([|\log \Phi_n| \geq M]) \leq \epsilon\ \forall\ \ n \geq n_\epsilon.$$

Sublemma 1.

$$\binom{n - \rho_n(x) - 1}{s_{n-\rho_n(x)-1}(x) - 1} \ \leq\ N_n(x)\ \leq$$
$$\leq \binom{n - \rho_n(x)}{s_{n-\rho_n(x)}(x)} + \binom{n}{s_n(x) + \rho_n(x) + \sigma_n(x) - 1}.$$

Proof. We first establish the lower bound. Letting

$$k_n(x) = 2^{n-\rho_n(x)} - \sum_{k=1}^{n-\rho_n(x)} 2^{k-1} x_k,$$

we see that

$$(\tau^{k_n(x)} x)_j = \begin{cases} 0 \quad 1 \leq j \leq n - \rho_n(x) - 1, \\ 1 \quad n - \rho_n(x) \leq j \leq n + \sigma_n(x) - 1, \\ x_k \quad \text{else.} \end{cases}$$

It follows that

$$N_n(x) \geq \# \left\{ k_n(x) \leq j \leq k_n(x) + 2^{n-\rho_n(x)-1} - 1 : \sum_{t=1}^{\infty}((\tau^j x)_t - x_t) = 0 \right\} =$$
$$= \# \left\{ (\epsilon_1, \ldots, \epsilon_{n-\rho_n(x)-1}) \in \{0,1\}^{n-\rho_n(x)-1} : \sum_{k=1}^{n-\rho_n(x)-1} \epsilon_k = s_{n-\rho_n(x)-1}(x) - 1 \right\}$$
$$= \binom{n - \rho_n(x) - 1}{s_{n-\rho_n(x)-1}(x) - 1}.$$

To check the upper bound, set $K_n(x) = k_n(x) + 2^{n-\rho_n(x)-1}$, and note that

$$\begin{aligned} N_n(x) &= \#\{0 \le j \le K_n(x) - 1 : \phi_j(x) = 0\} + \\ &\quad + \#\{K_n(x) \le j \le 2^n - 1 : \phi_j(x) = 0\} \\ &\le \#\{\underline{\epsilon} \in \{0,1\}^{n-\rho_n(x)} : s_{n-\rho_n(x)}(\underline{\epsilon}) = s_{n-\rho_n(x)}(x)\} \\ &\quad + \#\{\underline{\epsilon} \in \{0,1\}^n : s_n(\underline{\epsilon}) = s_n(x) + \rho_n(x) + \sigma_n(x) - 1\} \\ &= \binom{n - \rho_n(x)}{s_{n-\rho_n(x)}(x)} + \binom{n}{s_n(x) + \rho_n(x) + \sigma_n(x) - 1}. \end{aligned}$$

□

Sublemma 2. *Suppose that* $0 \le k \le n$, *and* $0 \le k + b \le n + a$, *then*

$$\left| \log \binom{n+a}{k+b} - \log \binom{n}{k} \right| \le$$

$$\le (|a| + |b|)\Big(|\log(p - \frac{|a| + |b|}{n})| + |\log(1 - p - \frac{|a| + |b|}{n})| \Big)$$

where $p := \frac{k}{n}$.

The proof of sublemma 2 is straightforward, and is left to the reader.

Proof of the main lemma. Define Φ_n by

$$N_n = \Phi_n \binom{n}{s_n}.$$

By sublemma 1,

$$N_n \ge \binom{n - \rho_n - 1}{s_{n-\rho_n-1} - 1}$$

and by sublemma 2,

$$\binom{n - \rho_n - 1}{s_{n-\rho_n-1} - 1} \ge [(p_n - \frac{a_n + b_n}{n})(1 - p_n - \frac{a_n + b_n}{n})]^{a_n+b_n} \binom{n}{s_n}$$

where $a_n = \rho_n + 1$, and $b_n = s_n - s_{n-\rho_n-1} + 1 \le \rho_n + 2$, whence

$$\Phi_n \ge [(p_n - \frac{2\rho_n + 3}{n})(1 - p_n - \frac{2\rho_n + 3}{n})]^{2\rho_n+3} \tag{5}$$

Again by sublemma 1,

$$N_n(x) \le \binom{n - \rho_n(x)}{s_{n-\rho_n(x)}(x)} + \binom{n}{s_n(x) + \rho_n(x) + \sigma_n(x) - 1},$$

and again by sublemma 2,

$$\binom{n-\rho_n}{s_{n-\rho_n}} \le \left[\frac{1}{(p_n - \frac{a_n+b_n}{n})(1-p_n-\frac{a_n+b_n}{n})}\right]^{a_n+b_n}\binom{n}{s_n}$$

where $a_n = \rho_n$, and $b_n = s_n - s_{n-\rho_n} \le \rho_n$,

$$\binom{n}{s_n(x)+\rho_n(x)+\sigma_n(x)-1} \le \left[\frac{1}{(p_n-\frac{b_n}{n})(1-p_n-\frac{b_n}{n})}\right]^{b_n}\binom{n}{s_n}$$

where $b_n = \sigma_n + \rho_n$, and it follows that

$$\Phi_n \le 2\left[\frac{1}{(p_n - \frac{2(\rho_n+\sigma_n)}{n})(1-p_n-\frac{2(\rho_n+\sigma_n)}{n})}\right]^{2(\rho_n+\sigma_n)}. \tag{6}$$

It follows from (5) and (6) that

$$|\log\Phi_n| \le$$
$$\le (2(\rho_n+\sigma_n)+3)\left|\log\left((p_n - \frac{2(\rho_n+\sigma_n)+3}{n})(1-p_n-\frac{2(\rho_n+\sigma_n)+3}{n})\right)\right|.$$

By the SLLN, μ_p-a.s.,

$$(p_n - \frac{2(\rho_n+\sigma_n)+3}{n})(1-p_n-\frac{2(\rho_n+\sigma_n)+3}{n}) \to p(1-p),$$

also,

$$(\rho_n+\sigma_n) = O(\log n),$$

whence

$$|\log\Phi_n| = O(\log n).$$

Also, given $\epsilon > 0$, if $K = 2|\log p(1-p)|$, and $p^{L-2} < \frac{\epsilon}{4}$, then,

$$\mu_p([2(\rho_n+\sigma_n)+3 \ge 2L+3]) \le \mu_p([\rho_n \ge L]) + \mu_p([\sigma_n \ge L]) \le 2p^{L-2} < \frac{\epsilon}{2},$$

and by the WLLN, for n large enough,

$$\mu_p\left(\left[\log((p_n - \frac{2(\rho_n+\sigma_n)+3}{n})(1-p_n-\frac{2(\rho_n+\sigma_n)+3}{n})) \ge K\right]\right) < \frac{\epsilon}{2}.$$

It follows that, for n large enough, $\mu_p([|\log\Phi_n| \ge K^{2L+3}]) < \epsilon$. □

4. Proofs of the Results

By Stirling's formula, and the SLLN, we have that

$$\binom{n}{s_n} \sim \frac{C_p}{\sqrt{n}} \frac{1}{p_n^{np_n}(1-p_n)^{n(1-p_n)}} = \frac{C_p}{\sqrt{n}} e^{nH(p_n)} \quad \mu_p - \text{ a.e. as } n \to \infty,$$

where

$$C_p = \frac{1}{\sqrt{2\pi p(1-p)}}, \text{ and } H(p) = -p\log p - (1-p)\log(1-p).$$

Combining this with the main lemma, we obtain that

$$N_n = \Psi_n \frac{e^{nH(p_n)}}{\sqrt{n}}, \tag{$*$}$$

where

$$|\log \Psi_n| = O(\log n) \quad \mu_p - \text{ a.e.},$$

and

$$\forall\, \epsilon > 0 \;\exists\; M = M_\epsilon,\ n_\epsilon \;\ni\; \mu_p([|\log \Psi_n| \geq M]) \leq \epsilon \;\forall\;\; n \geq n_\epsilon.$$

Proof of theorem 1. It follows from $(*)$ that

$$\frac{\log_2 N_n}{n} = H(p_n) + O(1) \to H(p) \text{ a.s. as } n \to \infty,$$

whence, since $N_n = S_{2^n}(1_\Omega)$,

$$\frac{\log S_n(1_\Omega)}{\log n} \to H(p) \text{ a.s. as } n \to \infty,$$

and theorem 1 follows from the ratio ergodic theorem. □

The other results are established by considering the Taylor expansion of H around p, and the asymptotic behaviour of $p_n - p$ as $n \to \infty$.

Let

$$s_n^* = s_n^{*,p} = \frac{s_n - np}{\sqrt{p(1-p)n}},$$

then

$$p_n - p = \sqrt{p(1-p)}\frac{s_n^*}{\sqrt{n}}.$$

By the central limit theorem (CLT),

$$\mu_p([s_n^* \geq \xi(\theta)]) \to \theta \ \forall\ 0 < \theta < 1,$$

and by the law of the iterated logarithm (LIL)

$$\varliminf_{n\to\infty} \frac{s_n^*}{\sqrt{\log^{(2)} n}} = -1, \ \varlimsup_{n\to\infty} \frac{s_n^*}{\sqrt{\log^{(2)} n}} = 1 \quad \mu_p - \text{ a.e..}$$

Expanding H around p, we obtain that

$$H(p_n) = H(p) + (p_n - p)H'(p) + \frac{(p_n - p)^2 H''(y)}{2}$$

for some y between p and p_n;

$$= H(p) + \log \frac{1-p}{p}\sqrt{p(1-p)}\frac{s_n^*}{\sqrt{n}} - \frac{p(1-p)}{2y(1-y)}\frac{s_n^{*2}}{n}.$$

Proof of theorem 2. It follows from the Taylor expansion of H around p, $(*)$ and LIL that

$$\begin{aligned}\log N_n &= nH(p_n) + O(\log n)\\ &= nH(p) + \log\frac{1-p}{p}\sqrt{p(1-p)n}s_n^* + O(\log n).\end{aligned} \tag{$\dagger$}$$

From ($\dagger$) and the CLT, we obtain that

$$\alpha_{2n}(m_p|_{\Omega\times\{0\}}, 1_{\Omega\times\{0\}}, \theta) = e^{nH(p)+c_p\sqrt{n}\xi(\theta)(1+o(1))}$$

as $n \to \infty$, whence

$$\alpha_n((m_p|_{\Omega\times\{0\}}, 1_{\Omega\times\{0\}}, \theta) = n^{\hat{H}(p)} e^{c_p\xi(\theta)\sqrt{\log n}(1+o(1))}$$

and (3) follows from lemma 1 of [**A3**].

To establish (4), choose $t \in \mathbb{R}$ and note that by ($\dagger$),

$$R(n,t) := \frac{N_n}{e^{nH(p)+t\sqrt{n\log^{(2)} n}}} = e^{\sqrt{n}\left(c_p s_n^* - t\sqrt{\log^{(2)} n}\right) + O(\log n)}$$

μ_p-a.e. as $n \to \infty$.

It now follows from LIL that

$$\varliminf_{n\to\infty} R(n,t) = \begin{cases} 0 & t > -c_p \\ \infty & t < -c_p \end{cases}, \ \& \ \varlimsup_{n\to\infty} R(n,t) = \begin{cases} 0 & t > c_p \\ \infty & t < c_p \end{cases}$$

Statement (4) follows from this and the ratio ergodic theorem. □

Proof of theorem 3. The proof of theorem 3 is slightly different.

To prove bounded rational ergodicity, we show that $\exists\ M > 0$ such that

$$S_n(1_{\Omega\times\{0\}}) \leq M \int_{\Omega\times\{0\}} S_n(1_{\Omega\times\{0\}})dm_{\frac{1}{2}}$$

for $n \geq 1$ and to obtain the return sequence, we show that

$$\int_{\Omega\times\{0\}} S_n(1_{\Omega\times\{0\}})dm_{\frac{1}{2}} \asymp \frac{n}{\sqrt{\log n}}.$$

These follow from

$$N_n \leq 2\binom{n}{[\frac{n}{2}]} \asymp \frac{2^n}{\sqrt{n}} \quad \text{and} \quad \varliminf_{n\to\infty} \frac{\sqrt{n}E(N_n)}{2^n} > 0,$$

which latter we prove.

By sublemma 1,

$$\begin{aligned} N_n(x) &\leq \binom{n-\rho_n(x)}{s_{n-\rho_n(x)}(x)} + \binom{n}{s_n(x)+\rho_n(x)+\sigma_n(x)-1} \\ &\leq \binom{n-\rho_n(x)}{[\frac{n-\rho_n(x)}{2}]} + \binom{n}{[\frac{n}{2}]} \\ &\leq 2\binom{n}{[\frac{n}{2}]}. \end{aligned}$$

To conclude, by $(*)$ and the Taylor expansion of H around $\frac{1}{2}$,

$$\begin{aligned} \log N_n &= nH(p_n) - \log\sqrt{n} + \log\Psi_n \\ &= n\log 2 - \log\sqrt{n} + \log\Psi_n - s_n^{*2} + o\Big(\frac{s_n^{*3}}{\sqrt{n}}\Big), \end{aligned}$$

whence $\liminf_{n\to\infty} \frac{\sqrt{n}E(N_n)}{2^n} > 0$. □

We conclude with the remark that there is no sequence of constants $a_n \to \infty$ such that

$$\frac{S_n^{T_{\frac{1}{2}}}(f)}{a_n}$$

converges in measure on sets of finite measure. If there were such a sequence, then for some $n_k \to \infty$,

$$a_{2^{n_k}} \propto \frac{2^{n_k}}{\sqrt{n_k}}$$

and

$$\log N_{n_k} - n_k \log 2 + \log \sqrt{n_k}$$

would converge in probability to a constant.

However

$$\log N_{n_k} - n_k \log 2 + \log \sqrt{n_k} = \log \Psi_{n_k} - s^{*2}_{n_k} + o\left(\frac{s^{*3}_{n_k}}{\sqrt{n_k}}\right)$$

whence by CLT,

$$\varliminf_{k\to\infty} \mu_{\frac{1}{2}}([\log N_{n_k} - n_k \log 2 + \log \sqrt{n_k} < -M]) > 0 \ \ \forall\, M > 0.$$

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